# Qualitative analysis of dissipative cosmologies

A. Di Prisco*
*Escuela de Física, Facultad de Ciencias,*
*Universidad Central de Venezuela, Caracas, Venezuela*

L. Herrera†
*Area de Física Teórica, Facultad de Ciencias,*
*Universidad de Salamanca, 37008 Salamanca, Spain*

J. Ibáñez‡
*Departamento de Física Teórica,*
*Universidad del País Vasco, Apdo. 644, 48080 Bilbao, Spain*
(February 7, 2008)

The evolution of an homogeneous and isotropic dissipative fluid is analyzed using dynamical systems techniques. The dissipation is driven by bulk viscous pressure and the truncated Israel-Stewart theory is used. Although almost all solutions inflate, we show that only few of them can be considered as physical solutions since the dominant energy condition is not satisfied.

98.80.Hw

## I. INTRODUCTION

There has been a renewed interest in the study of dissipative cosmologies related to the existence of inflationary solutions. As a matter of fact, dissipative processes are thought to be present in any realistic theory of the evolution of the universe. The simplest model of dissipative fluid is that which assumes the existence of bulk viscosity only. At a phenomenological level, bulk viscosity may be associated with particle production [1], [2]. Also, qualitatively, bulk viscosity may be understood as the macroscopic expression of microscopic frictional effects that appear in mixtures [3]. The easiest way to include bulk viscosity effects is through the Eckart's theory [4], which assumes that the bulk viscous pressure is proportional to the expansion. Several authors have used that theory to investigate the effects of viscosity on cosmological models [5].

As it is well known, however, Eckart's theory suffers important drawbacks [6] and a more complete theory must be used. One of the proposed theories is the Israel-Stewart theory [7] which complies with causality and stability. The non-linear terms of that theory are often neglected giving place to the so called truncated theory. Although the truncated theory may lead to a different behaviour as compared with the full theory, we shall for simplicity base our analysis on the truncated version of the Israel-Stewart approach. This truncated teory was firstly used by Belinskii et al. [8] in a cosmological context and after that by many authors ( see [9] and references therein). The flat FRW models were studied in ref. [10] and has been extended recently to spatially curved solutions [11]. In these last works the system of field equations was reduced to a second order differential equation and the stability of the stationary solutions was investigated by using a Lyapunov function. A different and more powerful approach was given by Coley and van den Hoogen [12] who used dynamical system techniques to analyze the asymptotic behaviour of the governing field equations. They used dimensionless equations of state that make the equation for the expansion to decouple from the system, obtaining a two dimensional system of differential equations.

Since these models were originally studied, it was soon realized that the type of equations of state used in them was determinant in verifying whether the models underwent inflation. It is therefore important to study the effect that other reasonable equations of state will produce in the qualitative behavior of the solutions.

In this paper we analyze the asymptotic behaviour of isotropic and spatially homogeneous models with a fluid with bulk viscosity. It has been pointed out [6] that although bulk viscosity is almost vanishing, in both the ultrarelativistic and the Newtonian regimes, its dynamical effects can not be neglected. On the other hand the evolution equations

---

*On leave
†e-mail: lherrera@gugu.usal.es
‡e-mail: wtpibmej@lg.ehu.es

given by this type of matter can be used to model another kind of sources (e.g. the string dominated universe described by Turok [13] or even a scalar field) The equations of state we assume are those introduced by Belinskii *et al.* [8]. Unlike the case studied in ref. [12], now the expansion does not decouple and we should deal with the complete three dimensional system, except for a particular value of the parameters. It is important to stress that the choice of the equations of state could dramatically change the behaviour of the model and, therefore, it is worthwhile to study the implications of different reasonable equations of state.

## II. FIELD EQUATIONS

Let us consider an homogeneous and isotropic spacetime with an imperfect fluid moving orthogonally to the hypersurfaces of constant curvature. The source is a fluid with bulk viscosity:

$$T_{ab} = \rho u_a u_b + (p + \sigma)(g_{ab} + u_a u_b), \tag{1}$$

where $\sigma$ is the viscous pressure $p$ is the thermodynamic pressure and $\rho$ is the energy density. The Einstein field equations and the conservation equations are written as:

$$\dot{\rho} = -3H(\rho + p + \sigma) \tag{2}$$

$$\dot{H} = -H^2 - \frac{1}{6}(\rho + 3p + 3\sigma) \tag{3}$$

$$3H^2 = \rho - \frac{3k}{a^2}, \qquad (k = \pm 1, 0), \tag{4}$$

where $a$ is the scale factor and $H \equiv \frac{\dot{a}}{a}$ is the Hubble parameter. We assume an evolution equation for the viscous pressure $\sigma$ given by the so called truncated Israel- Stewart theory (see [14] and references therein):

$$\sigma + \tau \dot{\sigma} = -3\xi H. \tag{5}$$

where $\xi$ is the coefficient of bulk viscosity and $\tau$ is the relaxation time $(\xi > 0, \tau > 0)$. Although eq.(5)(strictly speaking) is valid only on the assumption that the fluid is close to equilibrium $(|\sigma| \ll p)$ we will assume, for simplicity, that it describes the evolution of the viscous pressure even for regimes far from equilibrium. A more complete description would require taking into account the full transport equation of the Israel-Stewart theory. We assume a barotropic equation of state $p = (\gamma - 1)\rho$, $0 < \gamma \leq 2$ and the following relations:

$$\xi = \alpha \rho^m, \qquad \tau = \frac{\xi}{\rho}, \quad m, \alpha > 0 \tag{6}$$

which were used by Belinskii et al. [8].

In order to study the system of equations we define adimensional variables and a new time:

$$\Omega \equiv \frac{\rho}{3H^2}, \qquad \Sigma \equiv \frac{\sigma}{H^2}, \qquad H(t)dt = d\tau. \tag{7}$$

Now the equations (2), (3) and (5) reduce to

$$\Omega' = (\Omega - 1)[\Omega(3\gamma - 2) + \Sigma] \tag{8}$$

$$\Sigma' = -9\Omega - \Sigma \left[ \frac{(3\Omega)^{1-m}}{\alpha} H^{1-2m} - 2 - (3\gamma - 2)\Omega - \Sigma \right] \tag{9}$$

$$H' = -H \left[ 1 + \frac{3\gamma - 2}{2}\Omega + \frac{1}{2}\Sigma \right], \tag{10}$$

where $'$ means derivative with respect to $\tau$. In addition to the above equations we have the Friedmann equation (4):

$$H^2(\Omega - 1) = \frac{k}{a^2}. \tag{11}$$

We will assume $H$ positive in the open and flat models, so the system (8)-(10) is well defined. This is not true in the closed case, for which $H$ could become zero. So in this case the system should be understood as describing the evolution of closed models at early times when the expansion is positive.

2

Before studying the qualitative behavior of the system it is important to consider the energy conditions. The strong energy condition (SEC) implies that $\rho + 3p_{eff} \geq 0$, where $p_{eff} = p + \sigma$. In terms of the new variables this condition is written as

$$(3\gamma - 2)\Omega + \Sigma \geq 0. \tag{12}$$

The violation of this condition implies that the scale factor accelerates and the solution inflates. The dominant energy condition (DEC) requires $\rho + p_{eff} \geq 0$ or:

$$3\gamma\Omega + \Sigma \geq 0. \tag{13}$$

Matter not verifying this condition is considered unphysical, futhermore, it seems that the violation of this condition could lead to the violation of the generalized second law of thermodynamics [2].

## III. QUALITATIVE BEHAVIOUR

The line $\Omega = 1$ is an invariant space and divides the phase space of the system into three invariant sets: $\Omega > 1$ corresponding to closed models, $\Omega < 1$ corresponding to open models and $\Omega = 1$ to flat FRW solutions. Besides these invariant spaces, $H = 0$ is another invariant set that splits again the phase space into another three invariant sets. However we do not consider the $H < 0$ space for we assume that in early times $H > 0$.

When $m = 1/2$ the equation (10) decouples from the two other equations. Therefore the system becomes a two dimensional system which can be studied much more easily. To investigate the qualitative features of the system we will start with this case and then with the complete system.

### A. $m = 1/2$

In this case the system reduces to a two dimensional system of the form:

$$\Omega' = (\Omega - 1)[\Omega(3\gamma - 2) + \Sigma] \tag{14}$$

$$\Sigma' = -9\Omega - \Sigma\left[\frac{\sqrt{3\Omega}}{\alpha} - 2 - (3\gamma - 2)\Omega - \Sigma\right] \tag{15}$$

There are four equilibrium points:

$$
\begin{aligned}
&P^{\pm} : \Omega = 1, \ \Sigma = \Sigma^{\pm} = \frac{1}{2}\left[\left(\frac{\sqrt{3}}{\alpha} - 3\gamma\right) \pm \sqrt{\left(\frac{\sqrt{3}}{\alpha} - 3\gamma\right)^2 + 36}\right] \\
&P_3 : \Omega = 0, \ \Sigma = 0, \\
&P_4 : \Omega = \Omega_0 \equiv \left(\frac{\alpha}{\sqrt{3}}\frac{6\gamma + 5}{3\gamma - 2}\right)^2, \ \Sigma = -(3\gamma - 2)\Omega, \quad \gamma > \frac{2}{3}.
\end{aligned}
\tag{16}
$$

The stability of these points is given by the sign of the eigenvalues of the matrix of the linearized system around each point. The point $P_3$ has two eigenvalues of different signs, so it is a saddle point. The two eigenvalues of the point $P^+$ are positive and, therefore, it is a source point. The point $P^-$ has its two eigenvalues negative provided $\alpha > \alpha_0$, where

$$\alpha_0 \equiv \frac{\sqrt{3}(3\gamma - 2)}{6\gamma + 5}, \tag{17}$$

being, therefore, an attractor. When $\alpha < \alpha_0$ this point becomes a saddle. The behaviour of points $P_4$ and $P^-$ are opposite: when $\alpha > \alpha_0$, $P_4$ is a saddle but is an attractor when $\alpha < \alpha_0$. Let us note that since $\alpha > 0$ the condition $\alpha < \alpha_0$ implies $\gamma > 2/3$ and $\Omega_0 < 1$. Therefore in this case the only attractor is the point $P_4$. When $\alpha = \alpha_0$ both points, $P_4$ and $P^-$ coincide.

The exact solutions corresponding to each points are the following: $P_3$ is the Milne universe for which $a(t) = t$. At the points $P^{\pm}$ we have

$$a(t) = a_0 t^{H_0}, \qquad \rho(t) = \frac{3H_0^2}{t^2}, \qquad \sigma(t) = \frac{\Sigma^{\pm} H_0^2}{t^2}, \qquad H_0 \equiv \frac{2}{3\gamma + \Sigma^{\pm}}. \tag{18}$$

At $P^+$ we have $H_0 < 1$ so the solution is a FRW non inflating model. At $P^-$ the solution inflates when $\alpha > \alpha_0$ ($H_0 > 1$) which is equivalent to say that the solution does not verify the strong energy condition (12). The solution at $P_4$ is:

$$a(t) = a_0 t, \qquad \rho(t) = 3\left(\frac{6\gamma + 5}{3\gamma - 2} \frac{\alpha}{\sqrt{3}}\right)^2 \frac{1}{t^2}, \qquad \sigma(t) = -\frac{3\gamma - 2}{3}\rho, \tag{19}$$

and the deceleration parameter $q = 0$.

It is important to check whether the dominat energy condition (13) is satisfied by the solutions. This condition is always satisfied at the points $P^+$ and $P_4$. It is easy to see that when the point $P^-$ is an attractor ($\alpha > \alpha_0$) then eq. (13) is verified as long as $\gamma > \alpha/\sqrt{3}$. When $P^-$ is a saddle, $\alpha < \alpha_0$ means that $\gamma > \alpha/\sqrt{3}$ and the DEC is verified at this point. In Table I the main results concerning points $P_4$ and $P^-$ are summarized.

To conclude with the description of these solutions we will compactify the phase space in order to analyse the behaviour at the infinity. To do this we introduce polar coordinates: $\Omega = \overline{r}\cos\theta$ and $\Sigma = \overline{r}\sin\theta$. To make $\overline{r}$ finite we define

$$r = \frac{\overline{r}}{1 + \overline{r}}, \tag{20}$$

and a new time $\overline{\tau}$:

$$\frac{d\tau}{d\overline{\tau}} = 1 - r. \tag{21}$$

With these definitions the system (18)-(19) can be written as

$$\frac{dr}{d\overline{\tau}} = (1 - r)\left\{ r^2[(3\gamma - 2)\cos\theta + \sin\theta] - \frac{r}{\alpha}\sqrt{3r(1-r)\cos\theta}\sin^2\theta \right.$$
$$\left. + (1 - r)[2r - 3\gamma r\cos^2\theta - 10r\cos\theta\sin\theta] \right\}$$
$$\frac{d\theta}{d\overline{\tau}} = (1 - r)[-9\cos^2\theta + \sin^2\theta + 3\gamma\cos\theta\sin\theta]$$
$$-\frac{1}{\alpha}\sqrt{3r(1-r)\cos\theta}\sin\theta\cos\theta. \tag{22}$$

We deduce from the above equations that the circle $r = 1$ consists of equilibrium points. Close to the circle we get

$$\frac{dr}{d\overline{\tau}} \sim (1 - r)r^2[(3\gamma - 2)\cos\theta + \sin\theta]. \tag{23}$$

When $3\gamma\Omega + \Sigma > 0$ the above derivative is positive and, then, the points in the circle are attractors, while when $3\gamma\Omega + \Sigma < 0$ the derivative is negative and the points are sources.

In Fig.1 we have plotted integral curves of the system (22) for different values of the parameters. The dashed curve represents the invariant space $\Omega = 1$. Points to the left of each vertical lines violate respectivaly each of the energy conditions. Curves that start or end at $\theta = \pm\pi/2$ are unphysical since $\Omega$ becomes negative and that means that at some time of their evolution the energy density is negative.

In Fig.1(a) the orbits starting at $P^+$ and going to the point $P^-$, which is the only attractor for these values of the parameters, enter into the region where the strong energy condition is violated, so the solutions start inflating. Since $P^-$ violates the DEC, the solutions enter into a region where this condition is violated being, therefore, unphysical. So, we see that in this case the only physical solutions are those that start at $P^+$ and go to infinity. These physical solutions do not inflate. In Fig.1(b) we have plotted orbits corresponding to values of the parameters such that the point $P_4$ is a saddle point. The main features in this case are similar to that showed in Fig.1(a). The main difference is that now, since the attractor $P^-$ verifies the DEC, then there are physical solutions ending at this point. Finally in Fig.1(c) we have plotted orbits corresponding to the case for which the attractor is the point $P_4$. What is remarkable in this case is that there are solutions that after undergoing inflation their final state is a universe with $\Omega < 1$ showing that the presence of a state of inflation in the universe does not imply necessarily that $\Omega = 1$ [15].

Let us notice from these figures that if we assume a solution starting close to the equilibrium regime, i.e. $|\sigma| \ll \rho$ or $|\Sigma| \sim 0$ then, at the end of the evolution, $|\Sigma|$ is of the same order than $\Omega$ which is out of the range of validity of the theory. This is the so called "runaway" solution [6] . There are however solutions that fulfil this requirement. If we take $\alpha \ll 1$ then $\Sigma^- \sim -3\sqrt{3}\alpha + O(\alpha^2)$ and, in this case, we can choose the parameters in such a way that $P^-$ is an attractor that verifies DEC and the physical pressure $p + \sigma$ of the solution is positive. So, solutions starting with $|\Sigma| \sim 0$ evolve towards $P^-$, being close to equilibrium

**B.** $m \neq 1/2$

In this case the evolution is described by the three-dimensional system (8)-(10). Although it is more difficult to visualize the evolution, the behaviour is similar to that described in the previous section. To further simplify the equations we define a new variable

$$h = H^{1-2m}. \tag{24}$$

The system (8)-(10) is cast into a simpler form

$$\Omega' = (\Omega - 1)[\Omega(3\gamma - 2) + \Sigma] \tag{25}$$

$$\Sigma' = -9\Omega - \Sigma \left[ \frac{(3\Omega)^{1-m}}{\alpha} h - 2 - (3\gamma - 2)\Omega - \Sigma \right] \tag{26}$$

$$h' = -(1-2m)h \left[ 1 + \frac{3\gamma - 2}{2}\Omega + \frac{1}{2}\Sigma \right], \tag{27}$$

The equilibrium points of the above system are:

$$P^{\pm} : \Omega = 1, \; h = 0, \; \Sigma^{\pm} = \frac{3}{2}\left(-\gamma \pm \sqrt{\gamma^2 + 4}\right)$$

$$P_3 : \Omega = 0, \; h = 0, \; \Sigma = 0, \tag{28}$$

$$P_4 : \Omega = 1 \; h = \frac{3^m \alpha}{\gamma}, \; \Sigma = -3\gamma.$$

In addition to these points there is another one, $P_5$, which is valid only when $m > 1/2$:

$$P_5 \; : \; \Omega = 1, \; H = 0\,(h = \infty), \; \Sigma = 0. \tag{29}$$

Let us note that when $m > 1/2$ the equilibrium points, except $P_4$ and $P_5$, have $H = \infty$.

As before, by looking at the eigenvalues of the linearized system we obtain the stability of these points. The points $P^{\pm}$ have eigenvalues

$$3\gamma - 2 + \Sigma^{\pm}, \qquad 3\gamma + 2\Sigma^{\pm}, \qquad -\frac{1-2m}{2}(3\gamma + \Sigma^{\pm}). \tag{30}$$

For $P^+$ the two first eigenvalues are positive and the third is negative when $m < 1/2$ and positive when $m > 1/2$. Thus, this point is a saddle point if $m < 1/2$ and a source if $m > 1/2$. For $P^-$ the two first eigenvalues are negative and the third is positive when $m < 1/2$ and negative when $m > 1/2$. Thus, this point is a saddle point if $m < 1/2$ but is an attractor if $m > 1/2$.

The eigenvalues of the point $P_3$ are:

$$\frac{1}{2}(4 - 3\gamma + 3\sqrt{\gamma + 4}), \qquad \frac{1}{2}(4 - 3\gamma - 3\sqrt{\gamma + 4}), \qquad -(1 - 2m). \tag{31}$$

The first eigenvalue is positive and the second negative. The third is either negative or positive depending on the value of $m$. Thus this point is a saddle point. Finally $P_4$ eigenvalues are

$$-2, \qquad \frac{3}{2\gamma}[-(\gamma^2 + 1) \pm \sqrt{(\gamma^2 + 1)^2 - 2(1 - 2m)}\,]. \tag{32}$$

If $m < 1/2$ the three above eigenvalues are negative and, then, $P_4$ is an attractor. But if $m > 1/2$ one of the eigenvalues is positive and $P_4$ turns to be a saddle point.

To analize the stability of the point $P_5$ we change to spherical coordinates: $\Omega = \bar{r}\sin\theta\cos\phi, \; \Sigma = \bar{r}\sin\theta\sin\phi, h = \bar{r}\cos\theta$ and we compactify the phase space by using the coordinate $r$ given by (20) and the time $\bar{\tau}$ defined by (21). Close to the point $P_5$, $(r \sim 1, \; \sin\theta \sim 1 - r, \; \phi = 0)$ we obtain:

$$\frac{dr}{d\bar{\tau}} \sim -\frac{1}{2}(1 - 2m)3\gamma(1 - r)^2. \tag{33}$$

The above expression is positive when $m > 1/2$ and therefore this point is an attractor.

5

The solution described by the point $P_4$ is given by

$$a(t) = a_0 e^{H_0 t}, \qquad \rho = 3H_0^2, \qquad \sigma = -3\gamma H_0^2, \qquad H_0 = \left(\frac{3^m \alpha}{\gamma}\right)^{1/(1-2m)}. \tag{34}$$

This de Sitter solution was already obtained by Barrow [2]. Since the rest of equilibrium points have $H = 0$ (or infinity depending on the value of $m$) it is not possible to get the corresponding exact solutions (except for the point $P_3$ that corresponds to the Milne solution) We can, however, obtain approximate solutions for some of the points. When $m > 1/2$ the point $P^+$ is a source. By linearizing the system (25)-(27) around this point we get

$$\tau \to -\infty, \qquad H \sim H_0 e^{-\lambda \tau}, \qquad \lambda = \frac{1}{2}(3\gamma + \Sigma^+), \tag{35}$$

where $H_0$ is a constant of integration. Integrating the other two variables and using (11) to change from $\tau$ to $t$ we obtain:

$$t \to 0, \qquad \rho \sim \frac{3}{(\lambda t)^2}, \qquad \sigma \sim \frac{\Sigma^+}{(\lambda t)^2}, \qquad a(t) \sim t^{1/\lambda}. \tag{36}$$

To see the behavior close to the point $P_5$ we can integrate eq.(27) and after a straightforward calculation we obtain

$$t \to \infty, \qquad H \sim \frac{2}{3\gamma t}. \tag{37}$$

This behaviour in the vicinity of the point $P_5$ was already obtained by Belinskii et al. [8] and by Barrow [2].

As to the energy conditions, it is easy to see that both are satisfied by the points $P^+$ and $P_3$ and are violated by the point $P^-$. The DEC is satisfied by $P_4$ and $P_5$ but SEC is not satisfied by $P_4$ and is satisfied by $P_5$ when $\gamma \geq 2/3$.

The behavior described above is summarized in Table II.

The compactification of the phase space in the case $m \neq 1/2$ is much more difficult than in the $m = 1/2$ case, making almost impossible to get a complete description of the system. However we can obtain the main features concerning the qualitative evolution of solutions from the information on the equilibrium points and the numerical integration of the system.

In Fig.2 we have plotted the projections of the orbits, obtained by numerical integration of the system (25)-(27), onto the plane $(\Omega, \Sigma)$ for different values of the parameters. The straight lines indicate whether the solutions fulfil each of the energy conditions. Points below each line do not satisfy the corresponding energy condition. As in the former case there are solutions that enter to or come from the $\Omega < 0$ region. All these solutions are unphysical.

Fig.2(a) corresponds to the case where the only attractor is the point $P_4$. Besides the solutions that end at infinity, there are solutions approaching point $P^+$, which is a saddle, then entering into an inflationary regime and as approaching the point $P^-$, which is a saddle as well, they become unphysical since the dominat energy condition is violated. The values of the parameter in Fig.2(b) are such that there are two attractors: points $P^-$ and $P_5$. Since in this case $P^-$ does not satisfy the DEC, the solutions going to this point become unphysical. However, solutions ending at $P_5$ could be physical. These last solutions could be of physical interest for they are the only solutions that could fulfil the requirement that the viscous pressure be much less than the equilibrium pressure during their evolution. Starting at points for which $\Sigma \sim 0$ they evolve towards solutions with $\Sigma = 0$.


## IV. CONCLUSIONS


In this paper we have analyzed the qualitative behaviour of a particular type of dissipative fluids in FRW universes using the truncated Israel-Stewart theory. There have been a previous [10] analysis of the same system and with the same equations of state but only for the zero curvature FRW universes that was thereafter extended to the general case [11]. In those analysis the system of equations was reduced to a second order differential equation whose stationary solutions were studied by using a Liapunov function. That method does not discriminate, however, between different equilibrium points. We have seen in this paper that when $m \neq 1/2$ almost all the equilibrium points have $H = 0$ and all of them were reduced to only one stationary solution in the mentioned papers.

An important feature that emerges from our analysis is that for most of the equilibrium points either the dominat energy condition or the positivity of the energy density are not satisfied. Solutions evolving towards these equilibrium points, therefore, should be ruled out as unphysical. That means that in order to obtain physical solutions we have to be very careful with both the initial conditions and the values of the parameters. With respect to inflation, we

see that, generically, the presence of bulk viscosity makes the solutions enter into an inflationary regime. Only few solutions do not inflate.

If we consider the fluid interpretation of our system of equations, we have seen that almost all the solutions evolve in such a way that they go outside the regime in which the theory is valid (i.e. when $|\sigma| \ll p$). The only exceptions are those solutions which evolve towards the point $P_5$ when $m > 1/2$ which describes an inflating universe when $\gamma < 2/3$ or solutions evolving towards $P^-$ with $m = 1/2$ and $\alpha$ very small.

As mentioned before we have used the truncated equation (5), instead of the more general transport equation derived from the Israel-Stewart theory. As it has been shown [14], this latter equation may be written as

$$\tau_* \dot{\sigma} + \sigma = -3\xi_* H \left[ 1 + \frac{1}{\gamma c_b^2} \left( \frac{\sigma}{\rho} \right)^2 \right] \tag{38}$$

with

$$c_b^2 = \frac{\xi}{(\rho + p)\tau} \tag{39}$$

and

$$\tau_* = \frac{\tau}{1 + 3\gamma\tau H} \; ; \; \xi_* = \frac{\xi}{1 + 3\gamma\tau H}. \tag{40}$$

Therefore, close to equilibrium ($\sigma << \rho$), equation (38) leads to the truncated equation (5) with reduced relaxation time ($\tau_*$) and bulk viscosity ($\xi_*$)

$$\tau_* \dot{\sigma} + \sigma = -3\xi_* H. \tag{41}$$

The ammount of reduction depending on the size of $\xi$ relative to $H$. Obviously if $\tau H << 1$, then $\tau_* \approx \tau$ and $\xi_* \approx \xi$. However if $\tau H$ is close to 1, the reduction could be significant. On the other hand, qualitative changes in the results, outside the quasi-equilibrium regime, are difficult to forecast without considering specific models.

It is important to realize that the system of equations studied in this paper, not only describes the evolution of a dissipative fluid with bulk viscosity, but also the evolution of a different kind of matter. It is well konwn, for instance, that when $m = 2/3$ the above equations give the evolution of a string dominated universe [13]. We shall see that this equivalence can be extended to a scalar field as well.

Indeed, a scalar field in both FRW and Bianchi type solutions is equivalent to a perfect fluid with $p$ and $\rho$ defined as:

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \qquad p = \frac{1}{2}\dot{\phi}^2 - V(\phi), \tag{42}$$

where $V$ is the scalar field potential. We can substitute in the above expression $p$ by $p + \sigma$. If at the same time we consider a barotropic equation of state between the energy density and the equilibrium pressure of the form $p = (\gamma - 1)\rho$ then we obtain that the viscous pressure is given by:

$$\sigma = \frac{2 - \gamma}{2}\dot{\phi}^2 - \gamma V. \tag{43}$$

Taking derivatives with respect time and substituting into the Klein-Gordon equation of the scalar field

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0 \tag{44}$$

we obtain that $\sigma$ satisfies an equation like (5) with $\tau$ and $\xi$ given by

$$\tau = \frac{1}{6H}, \qquad \xi = \frac{1}{3H} \left( \gamma V + \frac{1}{3H}\frac{dV}{d\phi}\dot{\phi} \right). \tag{45}$$

Now, in dissipative systems there exists a peculiar state called "critical point", which corresponds to a specific value of a combination of thermodynamic variables. It has been shown [16], [17] (and references therein) that a dissipative system in the critical point immediately (i.e. on a time scale of the order of the relaxation time) after its departure from equilibrium, behaves as if the effective inertial mass of any fluid element vanishes.

In the case of pure bulk viscosity the critical point is characterized by

$$\frac{\xi}{\tau} = 2(\rho + p). \tag{46}$$

In this case however the critical point is forbidden by causality and stability requirements, demanding

$$\frac{\xi}{\tau} < (\rho + p)\left(1 - c_s^2\right) \tag{47}$$

where $c_s$ is the speed of sound (see [16]).

Nevertheless in order to see what the critical point may imply in terms of a scalar field, let us assume (46). Then, combining with (45), we obtain after an elementary algebra

$$\frac{1}{3H}\frac{dV}{d\phi} = \frac{\gamma\dot{\phi}}{2} \tag{48}$$

and feeding back into (44)

$$\ddot{\phi} + 3H\dot{\phi}\left(1 + \frac{\gamma}{2}\right) = 0. \tag{49}$$

A simple integration of (49) gives

$$\dot{\phi} \sim e^{-\int 3H(1+\gamma/2)dt}. \tag{50}$$

Let us consider two examples:

- de Sitter case, $H =$constant.

  In this well known case, one gets from (44)

  $$\dot{\phi} \sim e^{-3H(1+\gamma/2)t} \tag{51}$$

  and therefore

  $$\phi \sim \dot{\phi}. \tag{52}$$

  Then (48) yields

  $$V \sim \phi^2. \tag{53}$$

  This kind of potential is one of the most favoured by particle theorists, since it describes renormalizable particle theory [18].

- $H \sim \frac{2}{3\gamma t}$

  The asymptotic behaviour close to the point $P_5$ described above (see eq.(37)), yields, using (50)

  $$\dot{\phi} \sim \phi^{(1+\gamma/2)} \tag{54}$$

  and therefore the corresponding potential is

  $$V \sim \phi^{\gamma+2}. \tag{55}$$

Thus, the critical point condition (46), leads, for an scalar field interpretation of the source term, to polinomial potentials (for the two examples considered).

It is evident, therefore, that the constraints imposed by the fluid interpretation can be droped if we consider the energy momentum tensor to correspond to another kind of matter (different from the fluid with bulk viscosity). In such a case the only warnings to bear in mind are those coming from the dominat energy condition requirement.

[1] Ya.B. Zeldovich, Sov.Phys.JETP Lett **12**, 307 (1970); B.L. Hu, Phys.Lett. A **90**, 375 (1982).

[2] J.D. Barrow, Nucl. Phys B **310**, 743 (1988).

[3] N. Udey and W. Israel, Mon.Not.R.Astron.Soc. **199**, 1137 (1982); W. Zimdahl *Mon.Not.R.Astron.Soc.* **280** 1239 (1996).

[4] C. Eckart, Phys. Rev. **58**, 919 (1940).

[5] G.L. Murphy, Phys. Rev. D **8**, 4231 (1973); V.A. Belinskii and I.M. Kahalatnikov, Sov. Phys.-JETP, **42**, 213 (1991); A.A Coley and R.J. van den Hoogen, J. Math. Phys. **35**, 4117 (1994); A. Burd and A.A. Coley, Class. Quantum Grav. **11**, 83 (1994).

[6] W.A. Hiscock and J. Salmonson, Phys. Rev. D **43**, 3249 (1991).

[7] W. Israel, Ann. Phys.(N.Y.) **110**, 310 (1976); W. Israel and J. Stewart, Ann. Phys. (N.Y.) **118**, 341 (1979).

[8] V.A. Belinskii, E.S. Nikomarov and I.M. Khalatnikov, Sov. Phys.-JETP **50**, 213 (1979).

[9] R. Maartens, Class.Quantum Grav. **12**, 1455 (1995).

[10] D. Pavón, J. Bafaluy and D. Jou, Class. Quantum Grav. **8**, 347 (1991); L.P. Chimento and A.S. Jakubi, Class.Quantum Grav. **10**, 2047 (1993).

[11] L.P. Chimento, A.S. Jakubi and D. Pavón *Stable inflationary dissipative cosmologies* preprint (2000).

[12] A.A. Coley and R.J. van den Hoogen, Class. Quantum Grav. **12**, 1977 (1995).

[13] N. Turok, Phys. Rev. Lett **60**, 549 (1988).

[14] R. Maartens *Casual Thermodynamics in Relativity* astro-ph/9609119.

[15] G.F.R. Ellis in *Gravitation, Proceedings of the Banff Summer Research Institute on Gravitation*, Eds. R. Mann and P. Wesson, World Scientific, Singapore 1991.

[16] L. Herrera and J. Martínez, Class.Quantum Grav. **15**, 407 (1998).

[17] L. Herrera and A. Di Prisco, Gen.Rel.Grav. **31**, 301 (1999).

[18] L.H. Ryder, *Quantum Field Theory*, (Cambridge University Press, Cambridge).

FIG. 1. Plots of the integral curves of the sytem when $m = 1/2$. Vertical axis represents the coordinate $r$ an the horizontal axis represents the $\theta$ coordinate. They range from 0 to 1 and from $-\pi/2$ to $\pi/2$ respectively. Dashed line represents the curve $\Omega = 1$. Points to the right of the SEC line verify the strong energy condition and points left don't. The same for DEC line, corresponding to the dominat energy condition. (a) In this case $\gamma = 1/3$ and $\alpha = 1$, ($\alpha > \alpha_0$, $\gamma < \alpha/\sqrt{3}$). (b) Now $\gamma = 2$ and $\alpha = 1$, ($\alpha > \alpha_0$) and (c) $\gamma = 2$ and $\alpha = 0.2$, ($\alpha < \alpha_0$)

FIG. 2. Projections of the integral curves of the system when $m \neq 1/2$. Points below the solid diagonal line does not verify the strong energy condition, while those below the dashed diagonal line does not verify the dominat energy condition. (a) In this case $\gamma = 1/3$, $m = 1/4$ and $\alpha = 1$. (b) Now $\gamma = 1/3$, $m = 0.7$ and $\alpha = 1$.

TABLE I. Qualitative properties of points $P^-$ and $P_4$

| | | $P_4$ | $P^-$ | | DEC |
|---|---|---|---|---|---|
| $\alpha > \alpha_0$ | $\gamma < 2/3$ | | Attractor | $\gamma > \alpha/\sqrt{3}$ | Yes |
| | | | | $\gamma < \alpha/\sqrt{3}$ | No |
| $\alpha > \alpha_0$ | $\gamma > 2/3$ | Saddle | Attractor | $\gamma > \alpha/\sqrt{3}$ | Yes |
| | | | | $\gamma < \alpha/\sqrt{3}$ | No |
| $\alpha < \alpha_0$ | $\gamma > 2/3$ | Attractor | Saddle | | Yes |

TABLE II. Qualitative properties of points equilibrium points

|  | $P^+$ | $P^-$ | $P_3$ | $P_4$ | $P_5$ |
|---|---|---|---|---|---|
| $m < 1/2$ | Saddle | Saddle | Saddle | Attractor | |
| $m > 1/2$ | Source | Attractor | Saddle | Saddle | Attractor |
| DEC | Yes | No | | Yes | Yes |
| SEC | Yes | No | | No | if $\gamma \geq 2/3$, Yes |